1 Propagation of Uncertainties and Errors

1.1 Law of Propagation of Uncertainty

Propagation of errors is used when mathematical operations are performed on one or more measurements. The goal is to find the mean error of the result given the mean error of the input values.

The uncertainty σ_z of a quantity $Z = f(x_1,\, x_2,\, x_3,\, ...,\, x_N)$ (Z is known as the output) that depends on N input quantities $x_1,\, x_2,\, x_3, ...,\, x_N$ is found from[1]

$$\sigma_Z^2 = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_i^2 + \sum_{i=1}^{N}\sum_{j\neq i}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\sigma_{ij} \tag{1.1}$$

where σ_i^2 is the variance of x_i and σ_{ij} is the covariance of x_i and x_j. For example, x_1 could be position x, x_2 could be position y, and x_3 time t.

If the input quantities are independent (as is often the case), then the covariance σ_{ij} is zero and the second term vanishes.

Use of equation 1.1 allows us to calculate the combined error for any computational combination of measured or input variables.

1.2 Applications of the Law of Propagation of Uncertainties

The result of a calculation where the input values have uncertainties will itself have an uncertainty. To quantify it, the following rules are provided which are derived using equation 1.1. Here the notation is $z = z(x, y, t\ldots)$ and we wish to calculate σ_z, given values for σ_x, σ_y σ_z. Note that x, y, and t are assumed to be independent, so the second term in equation 1.1 is zero. The following rules calculate invalid results when x, y, and t are not independent.

1.2.1 Sum or Difference

$$z = x + y \qquad \sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2}$$

[1] An Introduction to Error Propagation: Derivation, Meaning and Examples of Equation $C_Y = F_X C_X F_X^T$ by Kai Oliver Arras, École Polytechnique. This can also be written as $\sigma_Z^2 = \sum_{i=1}^{N}\left(\frac{\partial f}{\partial x_i}\right)^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j=i+1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\sigma_{ij}$. The factor of 2 comes from the cross terms. Note how the i and j subscripting has changed.

$$z = x - y \qquad \sigma = \sqrt{\sigma_x^2 + \sigma_y^2}$$

1.2.2 Multiplication and Division

$$z = xy \qquad \sigma_z = |z|\sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

$$z = \frac{x}{y} \qquad \sigma_z = |z|\sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$

1.2.3 Multiplication and Division by an exact value c

$$z = c \cdot x \qquad \sigma_z = |c| \cdot \sigma_x \qquad\qquad z = \frac{x}{c} \qquad \sigma_z = \frac{\sigma_x}{c}$$

Example: $z = x + y - 3t,\ \sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2 + (3\sigma_t)^2}$

1.2.4 Exact Power

Here the value of x is not independent of itself in, for example, $(x \pm \sigma_x)^2$.

$$z = x^n \qquad \sigma_z = |z|\left(|n|\frac{\sigma_x}{|x|}\right)$$

Example: $z = \frac{xy}{t^3} \qquad \sigma_z = |z|\sqrt{\left(\frac{\sigma_x}{|x|}\right)^2 + \left(\frac{\sigma_y}{|y|}\right)^2 + \left(\frac{3\sigma_t}{t}\right)^2}$

1.2.5 Special functions

1.2.5.1 Functions of One Variable $z = f(x)$

Equation 1.1 will now have $N = 1$, so $\sigma_z^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_i^2$ which can be stated as $\sigma_z = \left(\frac{\partial f}{\partial x}\right)\sigma_i$ after taking the square root of both sides. The following examples are developed using this formulation.

$$z = \sin(x) \qquad \sigma_z = |\cos x|\sigma_x \qquad \text{for } \sigma_x \text{in radians}$$

$$z = \cos(x) \qquad \sigma_z = |-\sin x|\sigma_x \qquad \text{for } \sigma_x \text{in radians}$$

$$z = \ln x \qquad \sigma_z = \frac{\sigma_x}{|x|}$$

$$z = \mathrm{e}^x \qquad \sigma_z = \mathrm{e}^x \cdot \sigma_x \qquad \text{and} \qquad z = e^{5x} \qquad \sigma_z = 5e^x\sigma_x$$

1.2.5.2 Functions of Two Variables

If x and y are independent, $z = f(x, y)$ and equation 1.1 becomes $\sigma_z = \sqrt{\left(\frac{\partial f}{\partial x}\sigma_x\right)^2 + \left(\frac{\partial f}{\partial y}\sigma_y\right)^2}$. Consider for example,

$$x_1 = x_0 + vt$$

where $x_0 = 8.2 \pm 0.2\,\text{cm}$, $v = -20.2 \pm 0.4\,\text{cm}$ and $t = 0.14 \pm 0.02\,\text{sec}$. First consider vt as that involves one mathematical operation:

$$\sigma_{vt} = vt\sqrt{\left(\frac{\sigma_v}{v}\right)^2 + \left(\frac{\sigma_t}{t}\right)^2} = |-2.828\,\text{cm}|\sqrt{\left(\frac{0.4\,\text{cm}}{-20.2\,\text{cm}}\right)^2 + \left(\frac{0.02\,\text{sec}}{0.14\,\text{sec}}\right)^2}$$

$$\sigma_{vt} = 0.4078\,\text{cm}$$

Now consider the addition of x_0 and vt:

$$x_1 \pm \sigma_{x1} = (8.2 \pm 0.2\,\text{cm}) + (-2.828 \pm 0.4078\,\text{cm})$$

$$x_1 = 5.4\,\text{cm} \pm \sqrt{(0.2\,\text{cm})^2 + (0.4078\,\text{cm})^2}$$

$$x_1 = 5.4 \pm 0.5\,\text{cm}$$

when properly rounded.