AC Circuits : RC, RL, RLC

Capacitor Circuits : C [farads]

For a current i_C charging a capacitor such as shown in Figure 1a, the instantaneous voltage across the capacitor is

$$v_C = q/C$$

where q is the amount of +ve or −ve charge on one of the capacitor plates as shown in Figure 1a.

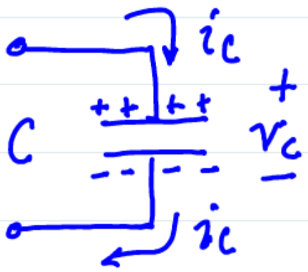


Figure 1a : v_C due to i_C flowing.

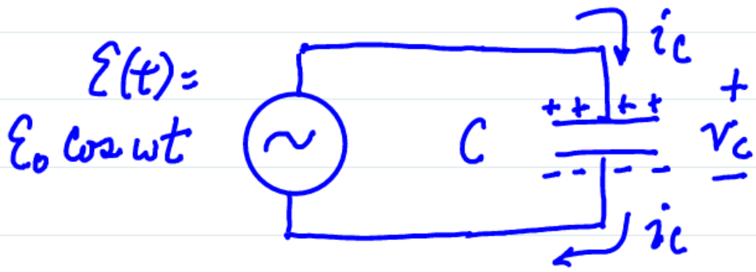


Figure 1b : AC source producing a current i_C.

If an AC source is in parallel with the capacitor as shown in Figure 1b, then

$$v_C(t) = V_C \cos \omega t$$

and $V_C = \mathcal{E}_0$, the amplitude of the AC source.

To calculate the current i_C, we can use the charge on the capacitor and differentiate it with respect to time :

$$q(t) = C v_C(t) = C V_C \cos \omega t$$

and

$$i_c = \frac{d}{dt}q(t) = \frac{d}{d}\left(C V_c \cos \omega t\right)$$

$$i_c = -\omega C V_c \sin \omega t \quad \text{------------------} \quad (1)$$

Since $-\sin \omega t = \cos(\omega t + \pi/2)$, the current can be written as

$$i_c(t) = \omega C V_c \cos(\omega t + \pi/2) \quad \text{-----------} \quad (2)$$

Thus the current created in C by $v_c(t)$ leads the applied voltage by $\pi/2$ or $90°$ as shown in Figure 2:

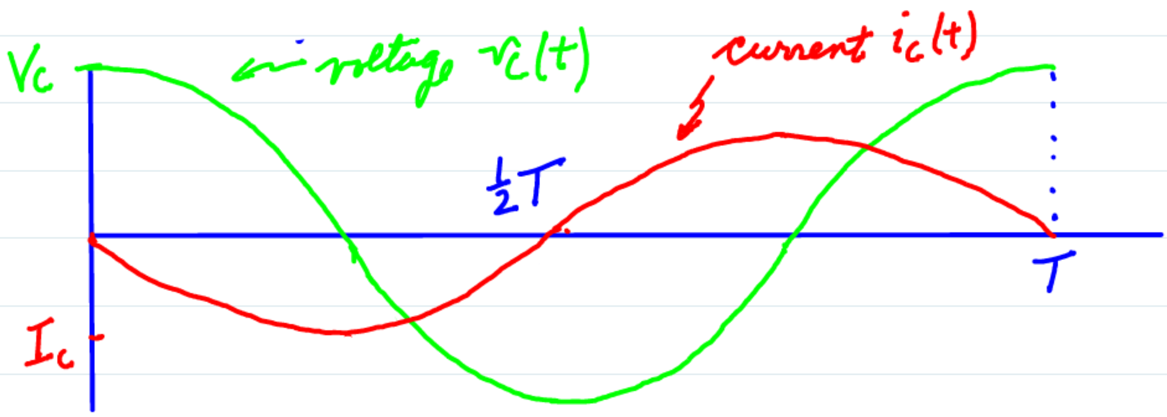


Figure 2: Current leading voltage for a capacitor circuit.

The term ωC in equation 2 represents the inverse of opposition to flow of charge (recall $i = v/R$) and is given the name reactance to distinguish it from resistance which has no frequency dependence:

$$X_c = \frac{1}{\omega C} \quad \text{--------------------} \quad (3)$$

This formulation of reactance shows the magnitude only. In order to account for phase shifts, X_C is represented by a vector in a complex plane.

magnitude: $|X_C| = \frac{1}{\omega C}$ and vector: $\vec{X}_C = \frac{-j}{\omega C} = \frac{1}{j\omega C}$

where $j = \sqrt{-1}$ and is imaginary.

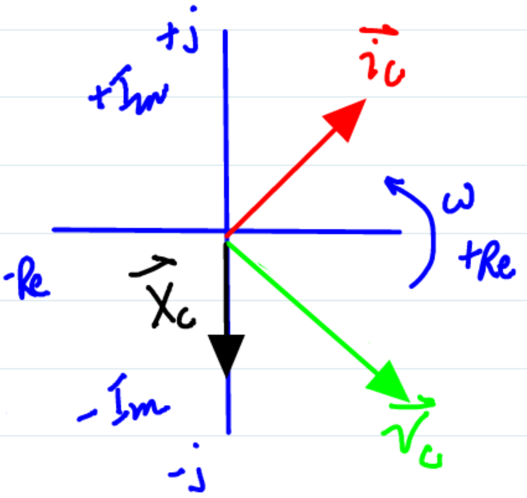


Figure 3: The i_C + v_C vectors rotate CCW at a rate of ω in radians per second. Note i_C leads v_C and $\vec{X}_C = -j/\omega C$. Since $\vec{X}_C$ has a set length, ω has been chosen.

Both the current and voltage waveforms can be represented on the complex plane as rotating vectors. The reactance vector does not rotate. Figure 3 shows this, time has been frozen. (Im = imaginary and Re = real axes.)

Capacitors are able to store charge; this ability equates to storing energy via the creation of an electric field between the two plates. The magnetic analog to this is the inductor.

Seeing that for a capacitor, the voltage lags the current (or current leads voltage), one can consider the capacitor to be a device which resists changes in the voltage across it. A sizeable current must flow for some amount of time before the voltage $v_C(t)$ begins to change.

Inductor Circuits: L [henries]

The instantaneous voltage across an inductor is related to the rate of change of current through the inductor by equation 4:

$$v_L(t) = L \frac{d}{dt} i_L(t) \text{ ---------------- } (4)$$

Using the same AC source as for the capacitor circuit, the voltage-current relationship can be determined:

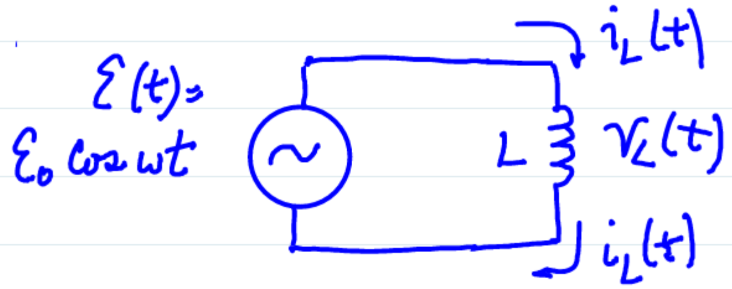


Figure 4: $v_L(t) = V_L \cos\omega t$ where $V_L = \mathcal{E}_0$.

Rearranging equation 4:

$$di_L = \frac{v_L}{L} dt$$

$$di_L = \frac{V_L}{L} \cos\omega t \; dt$$

Integrating gives: $i_L(t) = \frac{V_L}{L} \int \cos\omega t \, dt$

$$i_L(t) = \frac{V_L}{\omega L} \sin\omega t + i_0$$

($i_0 = \emptyset$ if no DC present)

Using $\sin\omega t = \cos(\omega t - \pi/2)$ lets one see the relationship between the voltage applied and the current through an inductor:

$$i_L(t) = \frac{V_L}{\omega L} \cos(\omega t - \pi/2) \text{ ------------- } (5)$$

Thus the current lags the voltage by 90° as shown in Figure 5:

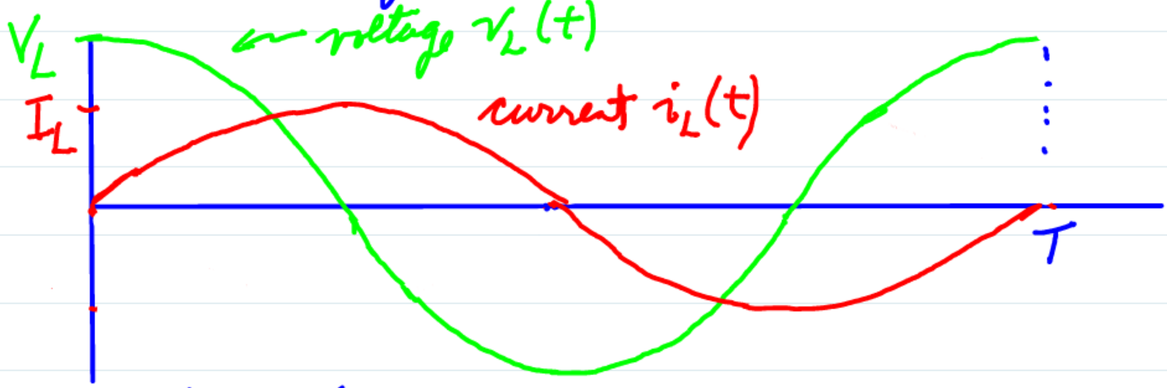


Figure 5: Current $i_L(t)$ lags applied voltage $v_L(t)$ for an inductor.

Figure 5 refers to the applied voltage, not the induced voltage on the inductor. The induced voltage will have a polarity which will oppose the change in current. The induced voltage is a reaction to a change in current*

The ωL factor in equation 6 is the inductor's reactance X_L in magnitude:

$$|X_L| = \omega L$$

As with the capacitor, the reactance is a vector quantity in the complex plane to allow for phase shifts between the voltage + current:

$$\vec{X_L} = j\omega L$$

The voltage and current vectors again rotate ccw as time passes as shown in Figure 6.

* a different situation or perspective

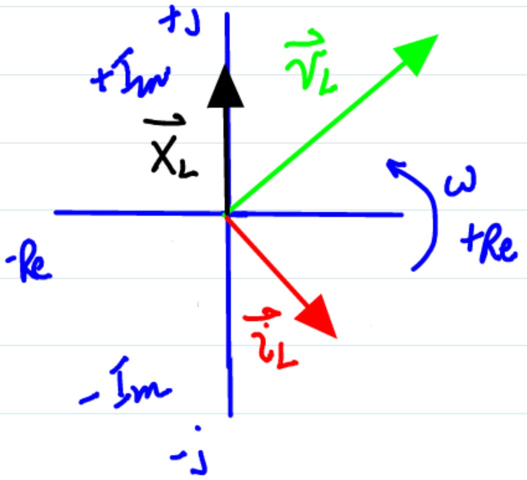


Figure 6: Current lags applied voltage for the inductor with reactance $\overline{X}_L = j\omega L$. Since $\vec{X}_L$ has a set length, ω has a set value.

Comparing the capacitive + inductive situations, this time the magnetic field is involved. In order for an applied voltage to generate a current in an inductor, a magnetic field must be created. This results in the "lagging" of the current. Similarly, once a current is established, energy is stored in the inductor in the form of the magnetic field present.

Farads + Henries:

The units of capacitance and inductance are farads and henries respectively. The farad can be considered the capacitance present in a pair of conductors sepearated by an insulator whose voltage rises to one volt if one coulomb of charge is present on one of the plates. The henry relates voltage to a change in current. If the current changes uniformly at a rate of one ampere per second and one volt is induced on an inductor, the inductance present is one henry (1 H). Both units are on the large end of the scale for practical capacitors and inductors: μF and mH values are common.

Frequency Dependence of Reactance

Both inductors and capacitors react to an applied sinusoidal voltage in a manner that shows a dependence on the angular frequency ω. For a capacitor, the reactance X_C varies from infinite for $\omega = \emptyset$ (a DC voltage) to zero as $\omega \to \infty$ due to its ω^{-1} dependence. This is shown in Figure 7a. The inductor's reactance scales linearly with ω as shown in Figure 7b.

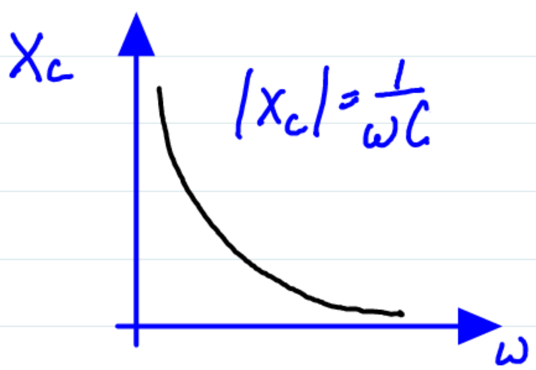


Figure 7a

The capacitor acts more + more like a short as $\omega \to \infty$. $|X_C| \to \infty$ as $\omega \to \emptyset$.

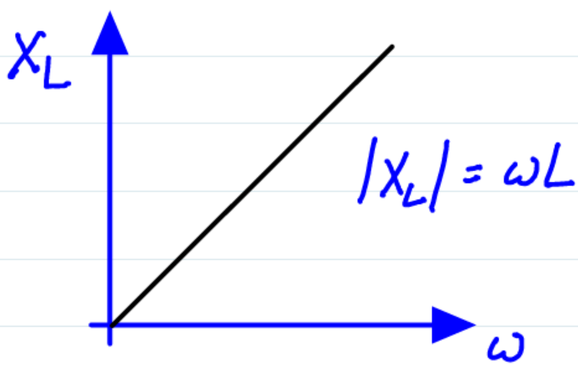


Figure 7b

The inductor acts like a short for $\omega = \emptyset$. $|X_L| \to \infty$ as $\omega \to \infty$

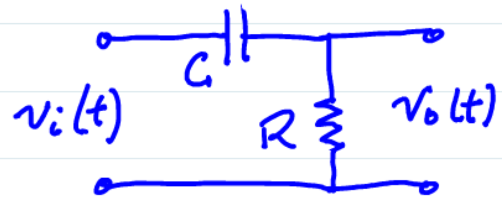
Both reactances are in units of ohms $[\Omega]$ and can be stated as "ohms reactive" to differentiate from the ohms of a resistor if needed.

Consequences of ω Dependence

With both devices having frequency

dependent reactances, both devices will filter or attenuate an applied voltage in a frequency dependent manner. Together with a series resistor, the RC or RL circuit can be a low or high pass filter depending on the order of the components (or where the output is taken). Both situations can be analyzed as a voltage divider with the reactance change accounted for. Each situation is illustrated below:

a)

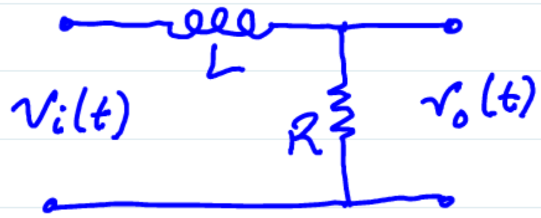


"High pass" $v_o(t) \to v_i(t)$ as $\omega \uparrow$ since $|X_C| \downarrow$ and $\phi \to 0°$

b)

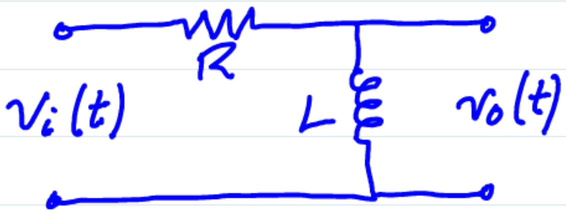
"Low pass" $v_o(t) \to \emptyset$ as $\omega \uparrow$ since $|X_C| \downarrow$, $\phi \to -90°$.
($v_o(t) \to v_i(t)$ as $\omega \to \emptyset$ and $\phi \to 0°$)

c)



"Low pass" $v_o(t) \to v_i(t)$ as $\omega \to \emptyset$ since $|X_L| \downarrow$ and $\phi \to \emptyset°$

d)



"High pass" $v_o(t) \to v_i(t)$ as $\omega \to \infty$ since $|X_L| \uparrow$ and $\phi \to 90°$. ($v_o(t) \to \emptyset$ as $\omega \to \emptyset$, $\phi \to 0°$)

Figure 8: As ω changes, $v_o(t)$ changes in amplitude and phase.

The Series RLC Circuit

Combining the three elements in series will produce a circuit showing resonant behavior analogous to a mechanical "systems" natural resonance. Strike an object and the vibration while have a dominant frequency. In electrical terms, applying a step voltage to an RLC circuit is similar to hitting an object. Applying a square wave is then the similar act of repeatedly hitting an object: beating the drum.

In terms of energy, resonance involves the capacitor and inductor exchanging energy. The $\vec{E}$ field in the capacitor grows and decays as the $\vec{B}$ field in the inductor decays and grows. The ω can be predicted as the frequency at which $|X_C| = |X_L|$. The frequency is denoted as ω_0:

$$1/\omega C = \omega L \ @\ \omega_0 \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

If measuring or setting the frequency in hertz:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \quad \text{-----------------} \quad (8)$$

where:

L is in henries and
C is in farads

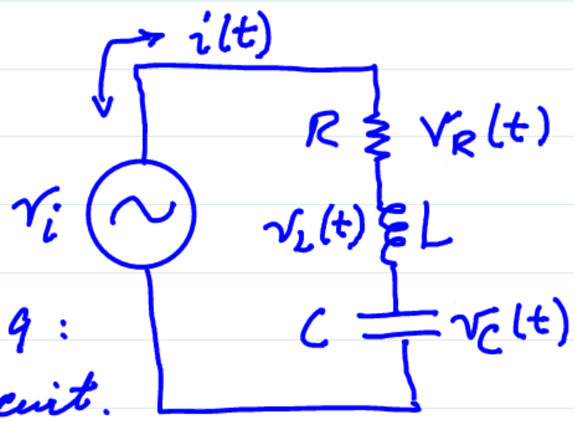


Figure 9: Series RLC circuit.

Exploring resonance in terms of the current, the resonant frequency for a series RLC circuit will produce the highest current. This occurs because the total impedance (symbol Z_{TOTAL} or Z_T) will be a minimum. Vector notation must be invoked. For the circuit in Figure 9:

$$\vec{Z_T} = \vec{R} + \vec{X_L} + \vec{X_C}$$

and using the complex plane (see Figure 10):

$$Z_T = R + j\omega L - j/\omega C$$

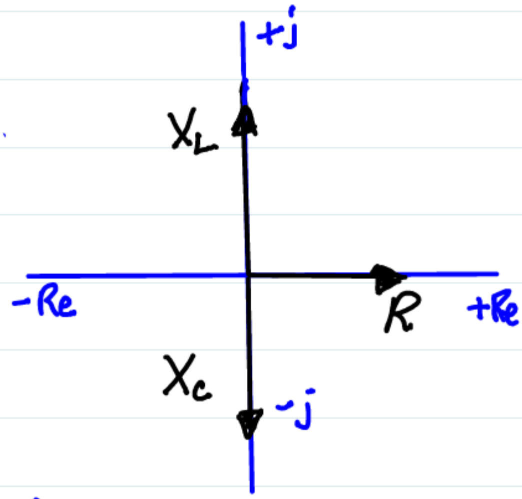


Figure 10: Impedance diagram @ resonance. X_L + X_C cancel when added.

At resonance $|X_C| = |X_L|$ so the total impedance Z_T is at minimum. Thus the condition of resonance can be determined (f_0) when $v_R(t)$ is at its maximum.
In addition, there will be zero
there will be no phase shift between the applied voltage and the current in the circuit.

Both the underlined conditions above are readily observable on an oscilloscope showing $v_R(t)$.

In addition if the circuits total resistance is low a ringing test can be performed by applying a square wave of sufficiently* short period.

* circuit dependent.

Experiment: RL Lowpass

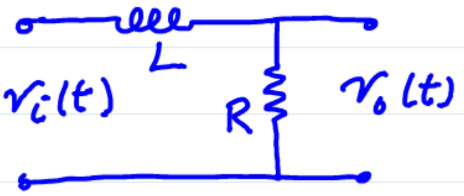


Figure 11

① Build the circuit shown in Figure 11. Use $L = 10\,mH$ and $R = 180\,\Omega$.

② Apply a $v_i(t)$ of 5 Vpp (sinusoidal waveform)

③ Make measurements of $v_o(t)$'s p-p amplitude and the phase shift of $v_o(t)$ relative to $v_i(t)$. Complete Table 1 for each frequency listed.

← measured →			← calculated →				
f [Hz]	v_o [pp]	$\phi°$ v_o w.r.t. v_i	$\lvert X_L \rvert\ \Omega$	$\frac{R}{(R^2+X_L^2)^{1/2}}$	v_o/v_i	$\theta° = \tan^{-1} \omega L/R$	
100							
200							
300							
500							
1 kHz							
2 kHz							
3 kHz							
5 kHz							
10 kHz							
20 kHz							
50 kHz							

Table 1: Data for RL circuit.

Q: Is v_o leading or lagging v_i?

Q: Is v_o in phase or out of phase with the current?

Experiment RC Highpass

Construct the circuit shown in Figure 12. Use $R = 680\,\Omega$ and $C = 0.1\,\mu F$. Make the measurements and calculations to complete Table 2.

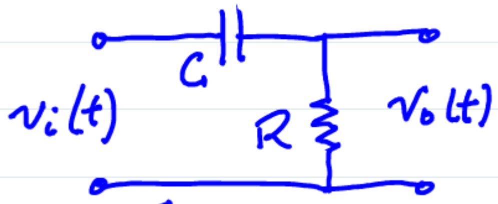


Figure 12

← measured → ← calculated →

f [Hz]	v_o [pp]	$\phi°$ v_o w.r.t. v_i	$\lvert X_C \rvert\ \Omega$	$\frac{R}{[R^2+X_C^2]^{1/2}}$	v_o/v_i	$90° - \tan^{-1}\omega RC$	
100							
200							
300							
500							
1 kHz							
2 kHz							
3 kHz							
5 kHz							
10 kHz							
20 kHz							
50 kHz							

Table 2 Data for RC circuit

Q: Is v_o leading or lagging v_i?

Q: Is v_o in phase or out of phase with the current?

Experiment: Series RLC Resonance

Measure the coil's resistance r_e and report these in a table similar to Table 3

Measure all L and C values.

2018 add R column

C	C measured	L	L measured	r_e
0.1 µF		1 mH		
1 µF		5 mH		
10 µF		10 mH		
100 µF		100 mH		

Table 3: Measured C, L + r_e

Build the circuit shown in Figure 13. Consult Table 4 for the values of L and C.

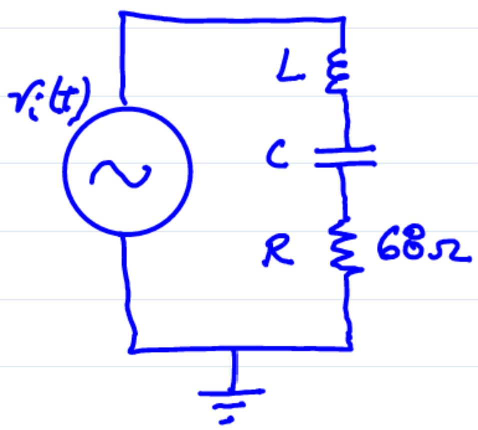


Figure 13: Series RLC

Use Channel 1 to monitor $v_i(t)$ and channel 2 to monitor $v_R(t)$. All scope and function generator grounds must be together at the ground location shown.

The signal generator's output should be set to 5 or 6 volts peak to peak.

Adjust the signal generator's sinusoidal output's frequency until the phase shift between channel 1 and channel 2 disappears

Record this frequency for the various combinations of L + C in Table 4!

Predict the resonant frequency using

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \quad [Hz]$$

Use the measured values, not the specification for the predictions

L [mH]	C [µF]	f_0 measured	f_0 predicted.
1mH	1µF		
5mH	1µF		
10mH	1µF		
100mH	1µF		
10mH	0.1µF		
10mH	10µF		
10mH	100µF		

Table 4: RLC resonance

Experiment: RLC Ringing Test

Change the series RLC circuit to that shown in Figure 14. Note R + C have moved.

2018: measure the 3.9nF capacitor: ______

Use $R = 10\Omega$, $L = 100mH$, $C = 3.9nF$ which is an orange color.

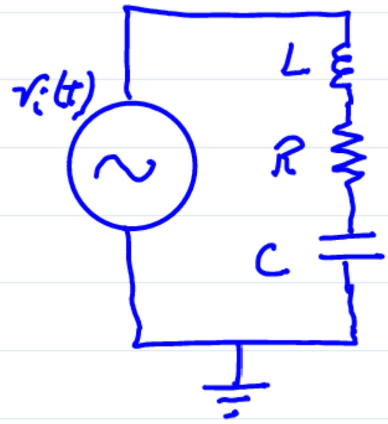


Figure 14: Circuit for Ringing Test.

Use channel 1 to view $v_i(t)$ and channel 2 to view $v_c(t)$

Adjust the signal generator's output to a square of around 600 Hz, 8 V_{pp}. Note these are approximate, further changes are likely required.

Monitor $v_i(t)$ with channel 1 and $v_c(t)$ with channel 2.

Once adjusted you will see a $v_c(t)$ similar to Figure 15.

Measure the period of the ringing waveform. This is the resonant frequency. See Figure 16.

Predict the resonant frequency using $f_0 = \frac{1}{2\pi\sqrt{LC}}$

Record the values in Table 5. Change L + C as needed.

Experiment: Dampening Effect

2018: change C back to 3.9nF.

Change R to 100 Ω and observe the ringing for several oscillations. That is shrink the scaling + signal generator frequency so many oscillations are shown.

Change R to 330 Ω.

Q: What effect does increasing R have on the ringing?

Q: What is ringing direct evidence of in terms of energy, electric field and magnetic field?

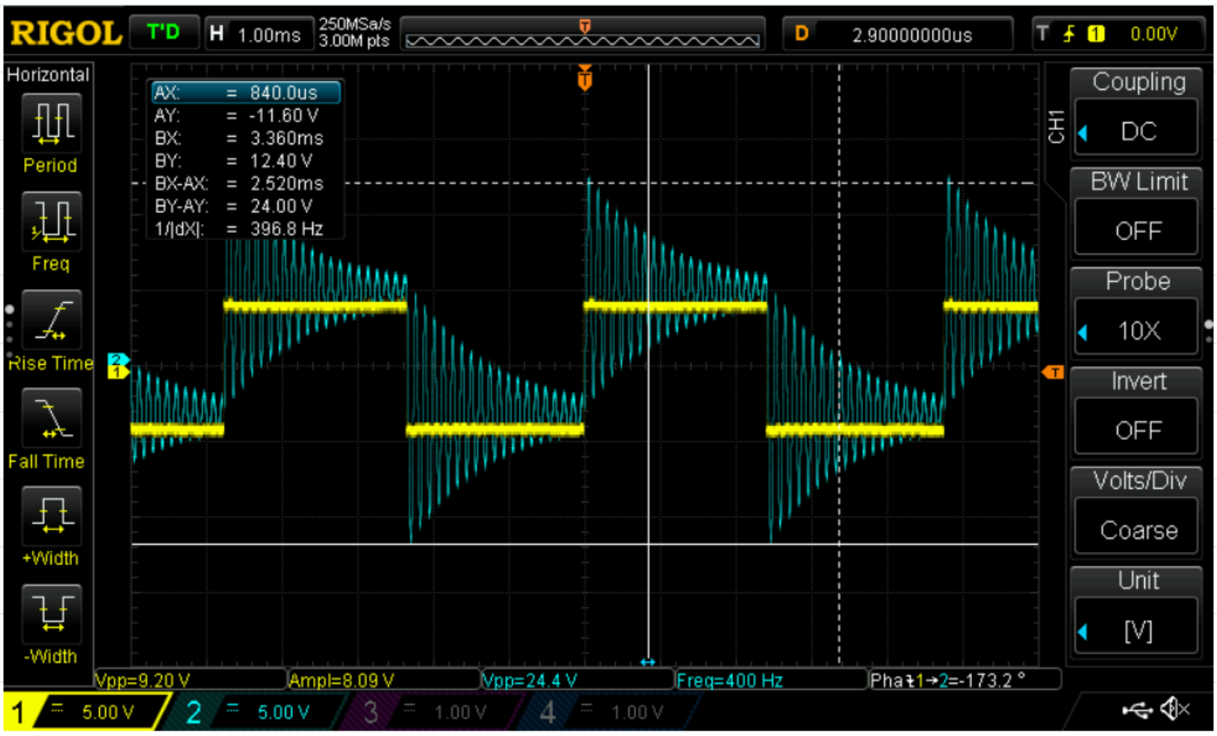


Figure 15: Ringing; several oscillations

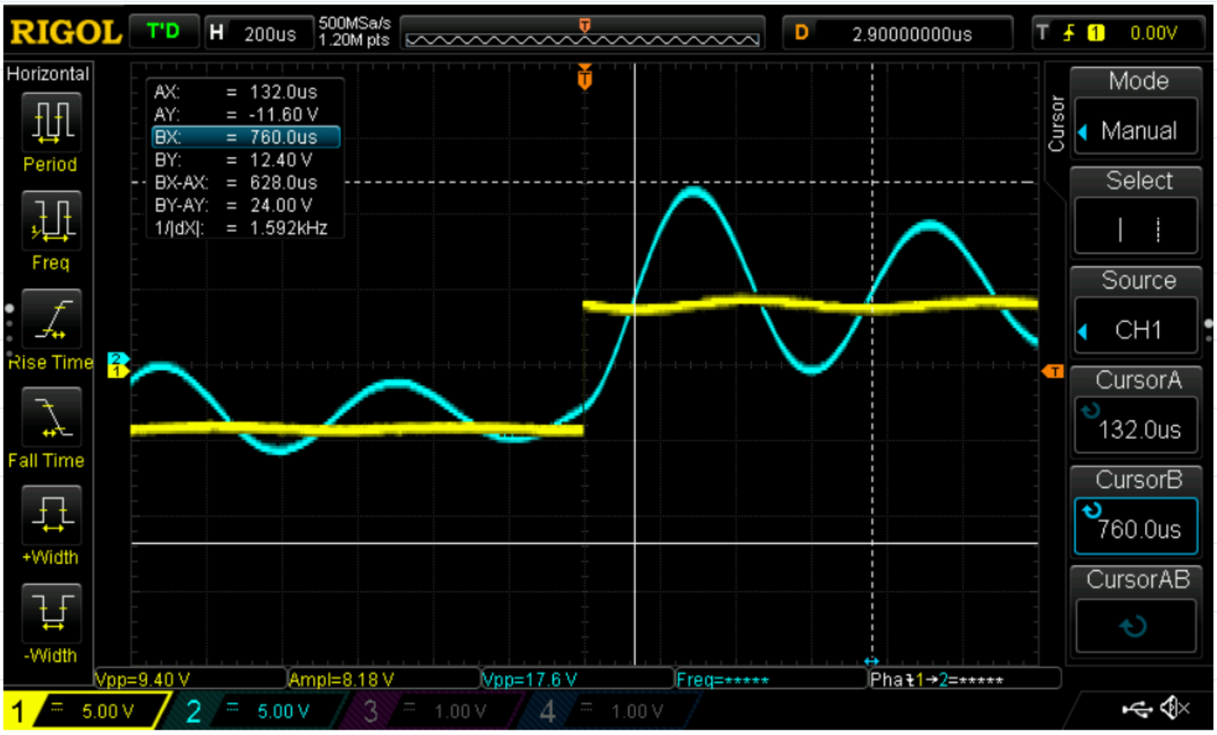


Figure 16: Ringing: Measuring the frequency.

L	C	R	T_{ring}	f_{ring}	f_o predicted
100 mH	3.9 nF	10 Ω			
100 mH	0.1 μF	10 Ω			

Table 5: Ringing Test Data